FMI NFA 2019-2020 - Homework 2

Ianis Vasilev, ianis@ivasilev.net

June 19, 2020

Exercise. ([Dei85, exercise 7.3]) Consider the subsets $B_2 \subseteq B_3 \subseteq B_1 \subseteq C([0,1])$, defined by

$$B_1 \coloneqq \left\{ x \in C([0,1]) \colon \begin{array}{r} 0 \leq t \leq 1 \implies 0 \leq x(t) \leq 1 \\ x(0) = 0, x(1) = 1 \end{array} \right\}$$

$$B_2 \coloneqq \left\{ x \in B_1 \colon \begin{array}{l} 0 \leq t \leq \frac{1}{2} \implies 0 \leq x(t) \leq \frac{1}{2} \\ \frac{1}{2} \leq t \leq 1 \implies \frac{1}{2} \leq x(t) \leq 1 \end{array} \right\}$$

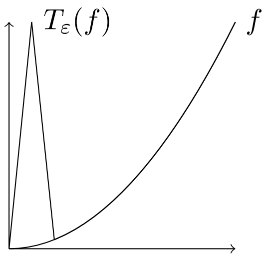
$$B_3 \coloneqq \left\{ x \in B_1 \colon \begin{array}{l} 0 \leq t \leq \frac{1}{2} \implies 0 \leq x(t) \leq \frac{2}{3} \\ \frac{1}{2} \leq t \leq 1 \implies \frac{1}{3} \leq x(t) \leq 1 \end{array} \right\}$$

Then $\alpha(B_1) = 1, \alpha(B_2) = \frac{1}{2}, \alpha(B_3) = \frac{2}{3}$ and $\beta(B_1) = \beta(B_2) = \beta(B_3) = \frac{1}{2}$.

Proof. Since the distance between any two functions from B_1 is at most 1, we have that $\operatorname{diam} B_1 = 1$ and $\alpha(B_1) \leq 1$.

Fix $\varepsilon > 0$. For any function $f \in B_1$, continuity of f gives us a radius $\delta_f > 0$ such that

$$x < 2\delta_f \implies f(x) < \varepsilon.$$



Define

$$T_\varepsilon(f)(x) := \begin{cases} \frac{x}{\delta_f}, & 0 \le x < \delta_f \\ f(\delta_f) + [1 - f(\delta_f)](2 - \frac{x}{\delta_f}), & \delta_f \le x < 2\delta_f \\ f(x), & x \ge 2\delta_f, \end{cases}$$

so that

$$\|T_\varepsilon(f) - f\| \ge T_\varepsilon(f)(\delta_f) - f(\delta_f) = 1 - f(\delta_f) > 1 - \varepsilon.$$

Additionally, because $\delta_{T_\varepsilon(f)} < \delta_f$, we have that $f(\delta_{T_\varepsilon(f)}) < \varepsilon$ and

$$\|T_\varepsilon(T_\varepsilon(f)) - f\| \ge T_\varepsilon(T_\varepsilon(f))(\delta_{T_\varepsilon(f)}) - f(\delta_{T_\varepsilon(f)}) = 1 - f(\delta_{T_\varepsilon(f)}) > 1 - \varepsilon.$$

Thus, proceeding by induction, we see that for any $m = 1, 2, \dots$

$$\|T_\varepsilon^m(f) - f\| > 1 - \varepsilon,$$

where T_ε^m denotes repeated application of T_ε.

Consider the sequence

$$\{T_\varepsilon^k(f)\}_{k=0}^\infty = \{f, T_\varepsilon(f), T_\varepsilon(T_\varepsilon(f)), \dots\}.$$

We can easily see that the distance between any two elements of the sequence, say $T_\varepsilon^k(f)$ and $T_\varepsilon^{k+m}(f)$, is strictly greater that $1 - \varepsilon$, i.e.

$$\left\|T_\varepsilon^k(f) - T_\varepsilon^{k+m}(f)\right\| = \left\|T_\varepsilon^k(f) - T_\varepsilon^m(T_\varepsilon^k(f))\right\| > 1 - \varepsilon.$$

Hence B_1 cannot be covered by a finite family of sets with diameter $1-\varepsilon$ and $\alpha(B_1) \ge 1 - \varepsilon$. Since $\varepsilon > 0$ can be made arbitrarily small, this implies that $\alpha(B_1) \ge 1$ and, because we already have the reverse inequality, $\alpha(B_1) = 1$.

In the set B_2, the maximum distance between two functions is $\frac{1}{2}$, thus $\operatorname{diam}(B_2) = \frac{1}{2}$ and $\alpha(B_2) \le \frac{1}{2}$. We can then define an operator similar to T_ε that creates “spikes” of height $\frac{1}{2}$ to prove the reverse inequality, obtaining

$$\alpha(B_2) = \frac{1}{2}.$$

Finally, the set B_3 has diameter $\frac{2}{3}$ and hence $\alpha(B_3) = \frac{2}{3}$.

The ball measure for B_1 satisfies the inequalities

$$\frac{1}{2} = \frac{\alpha(B_1)}{2} \le \beta(B_1) \le \alpha(B_1) = 1. \tag{1}$$

Additionally, B_1 is strictly contained in the ball centered in the constant function $\frac{1}{2}$ with radius $\frac{1}{2}$, which implies that $\beta(B_1) \le \frac{1}{2}$. Combining this with (1), we obtain $\beta(B_1) = \frac{1}{2}$.

For B_2 we have

$$\frac{1}{4} = \frac{\alpha(B_1)}{2} \le \beta(B_2) \le \alpha(B_2) = \frac{1}{2}. \tag{2}$$

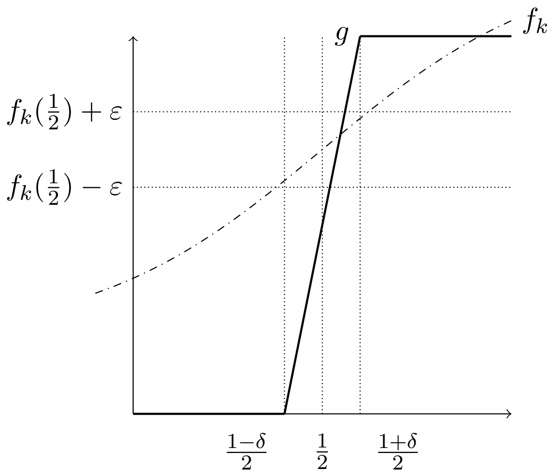
Assume that for some $\varepsilon > 0$ the set B_2 can be covered by a finite set of balls with centers $\{f_1, \dots, f_n\} \subsetneq C([0,1])$ and radius $\frac{1}{2} - \varepsilon$.

Because of continuity, we can find a radius $\delta > 0$ such that for all $f_k, k = 1, \dots, n$ we have

$$x \in \left[\tfrac{1-\delta}{2}, \tfrac{1+\delta}{2}\right] \implies \left|f_k(x) - f_k(\tfrac{1}{2})\right| < \varepsilon.$$

Consider the function

$$g(x) := \begin{cases} 0, & 0 \le x < \frac{1-\delta}{2}, \\ \frac{2x+\delta-1}{2\delta}, & \frac{1-\delta}{2} \le x \le \frac{1+\delta}{2}, \\ 1, & \frac{1+\delta}{2} < x \le 1. \end{cases}$$



If $f_k(\frac{1}{2}) \ge \frac{1}{2}$, then $f_k(\frac{1-\delta}{2}) > \frac{1}{2} - \varepsilon$ and

$$\|f_k - g\| \ge f_k(\tfrac{1-\delta}{2}) - g(\tfrac{1-\delta}{2}) = f_k(\tfrac{1-\delta}{2}) > \tfrac{1}{2} - \varepsilon.$$

Analogously, if $f_k(\frac{1}{2}) < \frac{1}{2}$, then $f_k(\frac{1+\delta}{2}) < \frac{1}{2} + \varepsilon$ and

$$\|g - f_k\| \ge g(\tfrac{1+\delta}{2}) - f_k(\tfrac{1+\delta}{2}) = 1 - f_k(\tfrac{1+\delta}{2}) > \tfrac{1}{2} - \varepsilon.$$

Thus, for every $k = 1, \dots, n$ we have

$$\|g - f_k\| > \frac{1}{2} - \varepsilon,$$

i.e. g in not contained in a ball of radius $\frac{1}{2} - \varepsilon$ around any of the centers $f_1, \ldots, f_n$.

Hence $\beta(B_2) \geq \frac{1}{2}$ and, because we already have the reverse inequality from (2), this implies $\beta(B_2) = \frac{1}{2}$. Because of the inclusion $B_2 \subsetneq B_3 \subsetneq B_1$, we have

$$\frac{1}{2} = \beta(B_2) \leq \beta(B_3) \leq \beta(B_1) = \frac{1}{2},$$

hence $\beta(B_3) = \frac{1}{2}$. □

References

[Dei85] Klaus Deimling. Nonlinear functional analysis. Springer-Verlag, 1985. ISBN: 0387139281.